## (2) Circulation

$$\Gamma \equiv \oint_C \vec{u} \cdot d\vec{s} = \int_A \vec{\omega} \cdot \vec{n}\, dA \quad \text{(Stokes' theorem)}$$

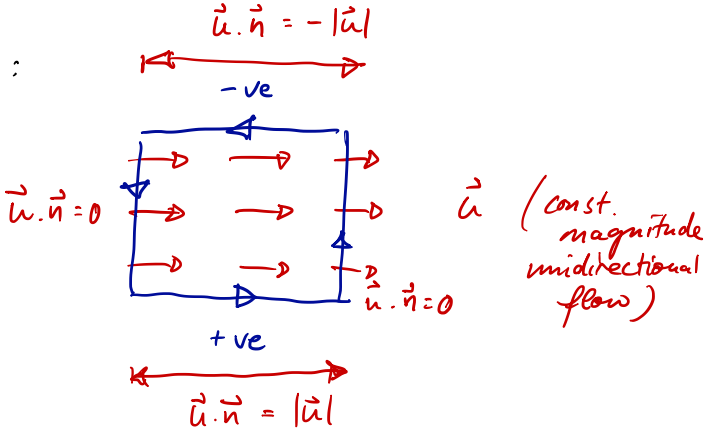
$\Gamma$ ← circulation

$$\frac{\Gamma}{A} = \lim_{A \to 0} \frac{1}{A} \int \vec{\omega} \cdot \vec{n}\, dA = \vec{\omega} \cdot \vec{n}$$

$\frac{\Gamma}{A}$ ← circulation per unit area

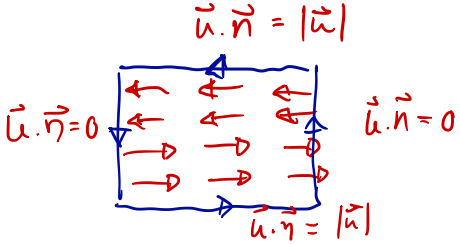
Vorticity ← $\left( \vec{n} \cdot \vec{\omega} = \lim_{A \to 0} \frac{1}{A} \oint \vec{u} \cdot d\vec{s} \right)$.

for example :




$$\oint \vec{u} \cdot d\vec{s} = -|\vec{u}| + |\vec{u}| + 0 + 0 = 0$$

↳ no circulation

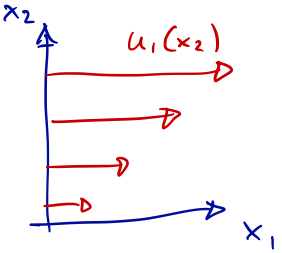



$$\oint \vec{u} \cdot d\vec{s} = |\vec{u}| + |\vec{u}| + 0 + 0 = 2|\vec{u}|$$

+ve $\Gamma$ (circulation)

# Simple Plane Flow

## Parallel shear flow : (steady)



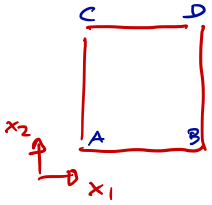

vel. gradient

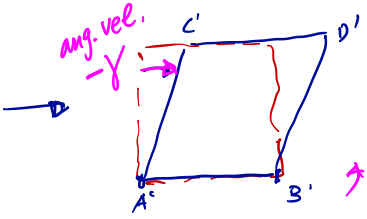
$$\vec{u} = \begin{bmatrix} u_1(x_2) \\ 0 \end{bmatrix} \qquad \frac{du_1}{dx_2} = \gamma(x_2)$$

Vorticity : $\vec{\nabla} \times \vec{u} = \vec{\omega} = (0, 0, -\gamma)$

ang. vel. $= -\gamma/2$

Consider Fluid element




avg. ang. vel. $\frac{1}{2}(-\gamma - 0) = -\gamma/2$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \therefore \quad S_{11} = S_{22} = 0$$

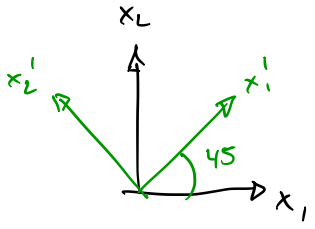
$$S_{12} = S_{21} = \gamma/2$$

$$\underline{\underline{S}} = \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix}$$

no elongation, pure shear

rotate coord. system

# Rotate coord. system.

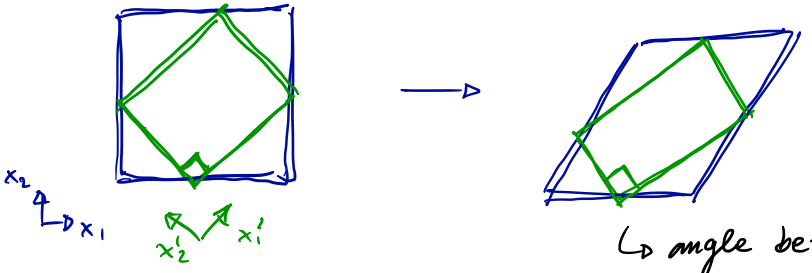



$$\underline{\underline{S}}' = \begin{bmatrix} \gamma/2 & 0 \\ 0 & -\gamma/2 \end{bmatrix}$$

(see tensor example for diagonalizing $S_{ij}$)

Note: $S_{ii} = 0 = S_{ii}'$

no area change in fluid element.




↳ angle between segments remains the same

∴ - no shear,
- pure elongation along $x_1'$
- pure contraction along $x_2'$

→ $\omega_3$ remains the same
(direction doesn't change, still out of $x_1$-$x_2$ plane)

$\omega_1'$ magn. is invariant!

($\vec{\omega}$ is a vector.)

# Steady Vortex Flows
## (circular streamlines)

① Solid body rotation

consider $(r, \theta)$ - polar coords.

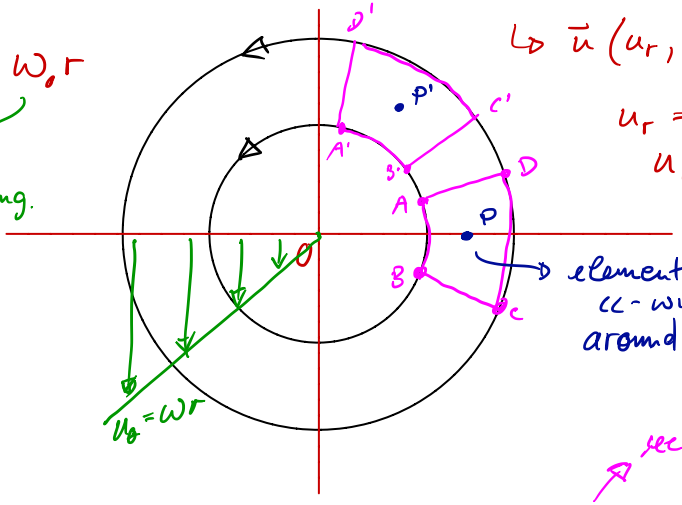
↳ $\vec{u}(u_r, u_\theta)$,

$u_r = 0$

$u_\theta = u_\theta(r)$

$u_\theta = \omega_0 r$

Const. ang. vel.




see HW Qu 2.

Vorticity : $\vec{\omega} = \vec{\nabla} \wedge \vec{u} = \vec{e}_z \left( \frac{1}{r} \frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} u_r \right)$

(no $\vec{e}_\theta$, $\vec{e}_r$ cpts)

$$\vec{\omega} = \vec{e}_z (2\omega_0) \quad \therefore \vec{\omega} = (0, 0, 2\omega_0)$$

→ remember $\vec{\omega}$ is twice the fluid rotation rate

∴ fluid elements rotate around P at rate $\omega_0$

∴ i.e. at same rate as every particle rotates around origin.

$$\underline{\underline{S}} = \begin{bmatrix} S_{rr} & S_{r\theta} \\ S_{r\theta} & S_{\theta\theta} \end{bmatrix} = 0 \quad \text{i.e. no deformation}$$

$u_\theta = \omega_0 r$

$$\underline{\underline{S}} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta} \\ & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \end{bmatrix}$$

Circulation around $R$ : $\Gamma = \oint \vec{u} \cdot d\vec{s} =$

$$= \int_0^{2\pi} \omega_0 R \, R \, d\theta = \omega_0 R^2 2\pi$$

$$\longrightarrow \Gamma = 2\omega_0 \times \pi R^2$$

$$= \text{vorticity} \times \text{area}$$

# Ideal Vortex



$\vec{u} = (u_\theta, u_r)$

(circular streamline without vorticity (except at origin))

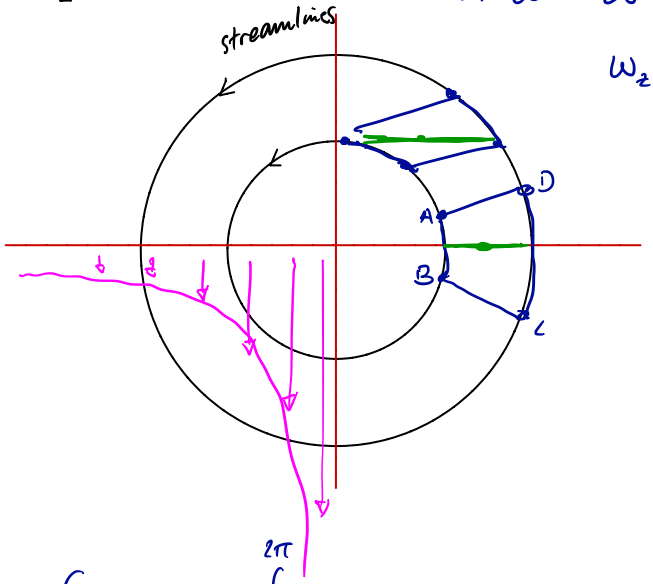
$$\vec{u} = \begin{bmatrix} c/r \\ 0 \end{bmatrix}$$

$\nabla \wedge \vec{u} = \vec{\omega} = 0$

$\omega_z = 0$ (everywhere except at origin)

↓

irrotational flow (except at origin)




$$\Gamma = \oint \vec{u} \cdot d\vec{s} = \int_0^{2\pi} \frac{c}{r} r\, d\theta = 2\pi c$$

↳ const.

$$[\omega_z]_{r\to 0} = \lim_{r\to 0} \frac{1}{A} \int \omega_z \, dA = \lim_{r\to 0} \frac{1}{\pi r^2} \int \vec{u} \cdot d\vec{s} = \lim_{r\to 0} \frac{2c}{r^2}$$

↓

vorticity → infinity

$\vec{u} \perp d\vec{s}$

$$\Gamma_{ABCD} = \left\{ \int_{AB} + \int_{BC}^{\vec{u}\perp d\vec{s}} + \int_{CD} + \int_{DA}^{\vec{u}\perp d\vec{s}} \right\} \vec{u} \cdot d\vec{s}$$

$$\Gamma_{ABCD} = -\Delta\theta \left[u_\theta r\right]_r + \Delta\theta \left[u_\theta r\right]_{r+\Delta r} = 0$$

∴ no circulation (already knew that since $\omega_z = 0$)

# Real Vortices

- bathtub
- wing-tips
- tornado

## Rankine vortex

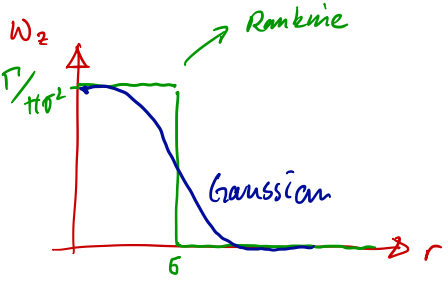
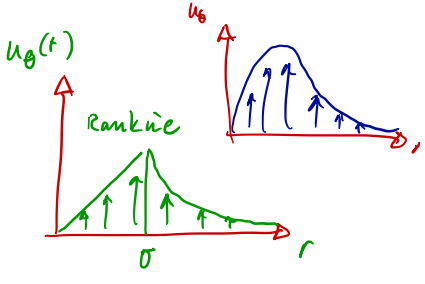
Solid body rotation

Abrupt transition at $r = \sigma$

$$\omega_z(r) = \begin{cases} \frac{\Gamma}{\pi\sigma^2} = \text{const.} & \text{for } r < \sigma \\ 0 & \text{for } r > \sigma \end{cases}$$

irrotational vortex

$$u_\theta(r) = \begin{cases} r\frac{\Gamma}{2\pi\sigma^2} & \text{for } r < \sigma \\ \frac{1}{r}\frac{\Gamma}{2\pi} & \text{for } r > 0 \end{cases}$$

## Gaussian vortex (gradual transition)

$$\omega_z(r) = \frac{\Gamma}{\pi\sigma^2} e^{-r^2/\sigma^2}, \qquad u_\theta(r) = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/\sigma^2}\right)$$