Description of `WE15_default.m` code

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This document is meant to accompany the code `WE15_default.m`, which is the model from the article

> *"How Climate Model Complexity Influences Sea Ice Stability"*,
> Wagner & Eisenman, Journal of Climate (2015).

The purpose of this document is to clarify the notation used in the code. In the following, equations are reproduced from the article with the equation numbers unchanged, and equations specific to this document are numbered as *Sxx*.

1 Governing Equations

The time evolution of $E(t,x)$ is determined at each latitude by the net energy flux:

$$\frac{\partial E}{\partial t} = \underbrace{aS}_{\text{solar}} - \underbrace{L}_{\text{OLR}} + \underbrace{D\nabla^2 T}_{\text{transport}} + \underbrace{F_b}_{\substack{\text{ocean}\\ \text{heating}}} + \underbrace{F}_{\text{forcing}}, \tag{2}$$

with insolation

$$S(t,x) = S_0 - S_1 \cos(\omega t)\, x - S_2 x^2, \tag{3}$$

co-albedo

$$a(x,E) = \begin{cases} a_0 - a_2 x^2, & E > 0 \quad \text{(open water)}, \\ a_i, & E < 0 \quad \text{(ice)}, \end{cases} \tag{4}$$

and OLR

$$L = A + B(T - T_m). \tag{5}$$

The temperature for freezing ice is given by solving

$$k(T_m - T_0)/h = -aS + A + B(T_0 - T_m) - D\nabla^2 T - F \tag{8}$$

for T_0.

We can define the temperature-independent and temperature-dependent components of the RHS of Eq. (2) as

$$\mathcal{C} = aS - A + BT_m + F, \qquad \mathcal{M} = B - D\nabla^2, \tag{S1}$$

such that

$$\frac{\partial E}{\partial t} = \mathcal{C} - \mathcal{M}T + F_b. \tag{S2}$$

2 Invoking a "ghost" layer to efficiently compute diffusion

Diffusion takes place in a "ghost" layer with temperature T_g that evolves according to

$$c_g \frac{\partial T_g}{\partial t} = \tfrac{c_g}{\tau_g}(T - T_g) + D\nabla^2 T_g, \tag{\textbf{A1}}$$

where c_g is the heat capacity of the ghost layer, and the first term on the right-hand side causes T_g to relax toward T with timescale τ_g. All other processes occur in the main layer, whose surface enthalpy evolves as

$$\frac{\partial E}{\partial t} = aS - A - B(T - T_m) - \frac{c_g}{\tau_g}(T - T_g) + F_b + F. \tag{\textbf{A2}}$$

We write this also in terms of temperature-independent and temperature-dependent components as

$$\frac{\partial E}{\partial t} = C - MT + F_b. \tag{S3}$$

with C and M (replacing $\mathcal{C}$, $\mathcal{M}$), defined as

$$C = aS - A + BT_m + \frac{c_g}{\tau_g} T_g + F, \quad M = B + \frac{c_g}{\tau_g} \tag{S4}$$

The temperature for freezing ice in the main layer, T_0, can be written as

$$k(T_m - T_0)/h = -C + MT_0, \tag{S5}$$

the solution of which (using $h = -E/L_f$) is

$$T_0 = \left(M - \tfrac{kL_f}{E}\right)^{-1} \left(C - \tfrac{kL_f}{E} T_m\right). \tag{S6}$$

3 Numerical Time-Stepping Scheme

Note that in the following we don't consider the spatial aspects of the integration (see Appendix A of the article for more detail). Here we are only concerned with the time stepping, but keep in mind that E, T, C are vectors of length n.

We time step the enthalpy using the Forward Euler method,

$$E_{i+1} = E_i + \Delta E_i, \tag{S7}$$

where

$$\Delta E_i = \Delta t(C_i - MT_i + F_b), \tag{S8}$$

with C_i being the forcing defined in Eq. (S4) evaluated at time $t_i = (i + 1/2)\Delta t$.

Next, to compute the evolution of T_g with the implicit Backward Euler method, we write

$$T_{g,i+1} = T_{g,i} + \Delta T_{g,i+1}, \tag{S9}$$

which we eventually want to solve for $T_{g,i+1}$ algebraically. Writing (**A1**) in discrete form, we have

$$\Delta T_{g,i+1} = \Delta t \left[\frac{1}{\tau_g} \left(T_{i+1} - T_{g,i+1} \right) + \frac{1}{c_g} D \nabla^2 T_{g,i+1} \right]. \tag{S10}$$

The issue here is that T is discontinuous at the ice edge and we'll have to account for the three regimes (freezing ice, melting ice, open water) separately. Some care needs to be taken since C_{i+1} depends on $T_{g,i+1}$. Note that we have

$$C_{i+1} = \mathcal{C}_{i+1} + \frac{c_g}{\tau_g} T_{g,i+1}, \tag{S11}$$

where $\mathcal{C}$ is defined in (S1). This gives (taking $T_m = 0$)

$$T_{i+1}(\text{water}) = \frac{E_{i+1}}{c_w}, \quad T_{i+1}(\text{melt}) = 0, \quad T_{i+1}(\text{freeze}) = \frac{\mathcal{C}_{i+1} + \frac{c_g}{\tau_g} T_{g,i+1}}{M - \frac{k L_f}{E_{i+1}}}, \tag{S12}$$

where (water), (melt), and (freeze) indicate grid boxes that are identified by $(E > 0)$, $(E < 0 \;\&\; T > 0)$, and $(E < 0 \;\&\; T < 0)$, respectively.

Substituting this into Eq. (S10) and solving for $T_{g,i+1}$ results in the following expression:

$$\begin{aligned} T_{g,i+1} = & \left\{ \kappa_{\text{all}} - \frac{\Delta t c_g}{\tau_g^2} \text{diag} \left(\frac{1}{M - \frac{k L_f}{E_{i+1}}} \right)_{\text{freeze}} \right\}^{-1} \\ & \times \left\{ T_{g,i} + \frac{\Delta t}{\tau_g} \left(\frac{E_{i+1}}{c_w} \right)_{\text{water}} + \frac{\Delta t}{\tau_g} \left(\frac{\mathcal{C}_{i+1}}{M - \frac{k L_f}{E_{i+1}}} \right)_{\text{freeze}} \right\}, \end{aligned} \tag{S13}$$

where

$$\kappa_{\text{all}} \equiv \left(1 + \frac{\Delta t}{\tau_g} \right) I - \frac{\Delta t}{c_g} D \nabla^2$$

is the only term that does not have to be computed at every timestep. All other terms need to be recomputed for each i.