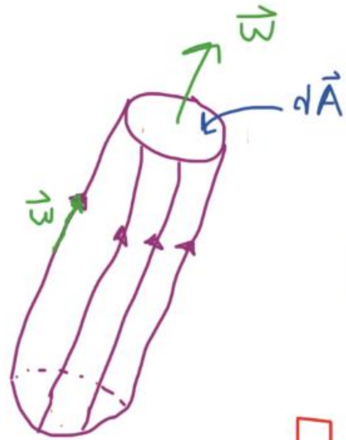
Vortex Tubes



$d\Gamma = \vec{\omega}\cdot d\vec{A}$

Γ : **Strength of the vortex tube**

$$\int_A \vec{\omega}\cdot\vec{n}\,dA = \left\{ \int_{top} + \int_{bottom} + \int_{side} \right\} \vec{\omega}\cdot\vec{n}\,dA$$

$$= \Gamma_{top} - \Gamma_{bottom} = 0^{*} = \int_V \vec{\nabla}\cdot\vec{\omega}\,dV$$

$$\vec{\nabla}\cdot\vec{\omega} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0$$

$$\Gamma_{top} = \Gamma_{bottom}$$

The strength of the vortex tube is the same everywhere, which implies vortex lines can't end within the fluid.

Solid body rotation

$$S_{ij}=0 \quad \therefore \; T_{ij}=-p\delta_{ij}$$

Cauchy's Formula

$$\rho\frac{Du}{Dt}=\rho g_j+\frac{\partial}{\partial x_i}\left(T_{ij}\right)$$

Euler's Formula

$$\rho\frac{Du}{Dt}=-\vec{\nabla}p+\rho\vec{g}$$

No deformation under solid body rotation
so strain rate = 0

Cauchy's formula simplifies to Euler's formula

$$u_r=0 \qquad u_\theta=\frac{\omega r}{2}$$

$$\rho u_\theta^2/r=\frac{\partial p}{\partial r} \quad \text{and} \quad 0=-\frac{\partial p}{\partial z}-\rho g$$

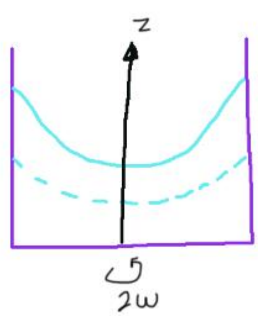
1) $\int \partial p=\int -\rho g\,\partial z$

$$\Rightarrow p(r,z)=-\rho g z+g(r)$$

2) $\int \partial p=\int \frac{\rho\omega^2 r}{4}\,\partial r$

$$\Rightarrow p(r,z)=\frac{\rho\omega^2r^2}{8}+f(z)$$

$$p(r,z)-P_0=\frac{\rho\omega^2r^2}{8}-\rho g z$$



$$\boxed{z=\frac{\omega^2r^2}{8g}-\frac{P(r,z)-P_0}{\rho g}}$$

Irrotational Vortex

Viscous Stress: $\tau_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right] = -\frac{\mu\Gamma}{\pi r^2}$

So viscous stress is nonzero everywhere, but viscous forces cancel on every fluid element except at r=0

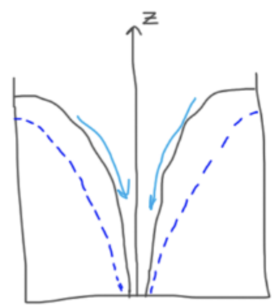
∴ Momentum again is described by Euler's equation

$$u_\theta = \Gamma / 2\pi r$$

$$p(r,z) - \underline{p_\infty} = -\frac{\rho\Gamma^2}{8\pi^2 r^2} - \rho g z$$

$$z = \frac{-\Gamma^2}{8\pi^2 r^2 g} - \frac{p(r,z) - p_\infty}{\rho g}$$

Second degree hyperboloids of rotation



Irrotationality does not imply the absence of viscous stresses, but it does imply the absence of net viscous forces.